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METHOD OF INITIAL FUNCTIONS FOR THICK SHELLS

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Abstract—In the present work for circular cylindrical shells, three-dimensional elasticity equations are solved by assuming Taylor series expansions, in the radial direction, for the stresses and displacements. Depending upon the number of terms retained in the expansion, different order shell theories are derived,, classical theories (referred to as eighth-order), the shear deformation-transverse normal stress theories (referred to as tenth-order), and higher order theories (referred to as twelfth-order). In each case, by carrying out the symbolic algebra using the digital computer, partial differential equations are derived. The procedure was carried out in detail for the case of a circular cylindrical shell with no loading on the interior surface and a given pressure distribution on the exterior surface. Then, numerical comparisons are made between the current theories and various shell theories, as well as the exact (three dimensional) theory. Thus, using this method with its associated computer programs, one can realize a spectrum of approximate shell theories ranging from the classical thin shell, through all current thick shell theories, right up to the three-dimensional elastic theories.

NOTATION

$\bar{r}, \theta, \bar{z}$ Cylindrical coordinates.
u, v, w Normalized displacement components.
r_0 Middle surface radius of cylinder.
a Inner radius of cylinder.
b Outer radius of cylinder.
r, z Normalized radial and axial coordinate.
h Thickness of shell.
$\mu = h/a$ Normalized thickness.
$\tau_{rz}, \tau_{r\theta}, \tau_{z\theta}$ Normalized shear stress components.
$\sigma_r, \sigma_\theta, \sigma_z$ Normalized normal stress components.

$$P = \frac{1}{1 - \nu}$$

$$\beta = \frac{\partial}{\partial \theta}$$

$$\alpha = a \frac{\partial}{\partial z}$$

$$a \frac{\partial}{\partial r}(\ ) = (\ )'$$

λ Exponential coefficient.

INTRODUCTION

In the present work, the method of initial functions has been used to solve the elasticity equations for isotropic materials by assuming Taylor series expansions for stresses and displacements in the radial direction at interior or exterior edges, for circular cylindrical shells. Comparison is then made with classical shell theories of different orders, as well as three-dimensional shell theory as a special case of elasticity theory.

The method of initial functions, in which the elasticity equations are solved by assuming Taylor series expansions for stresses and displacements was introduced by Vlasov[1] for elasto-static problems. Das and Setlur[2] applied the method to two-dimensional elasto-dynamic problems both in plane stress and plane strain in which Maclaurin series were expanded in the thickness direction.

Haydl[3] used Vlasov's method of initial parameters for symmetric bending of cylindrical shells, and by an example he showed his method is applicable for the solution of shells with many supports or concentrated loading.

Iyengar *et. al*[4] used this method for investigating thick rectangular plates, for the case of two opposite edges simply supported and the other two edges clamped.

Das and Rao[5] applied this method to thick plates subjected to antisymmetric and symmetric lateral loads.

This method was extended by Iyengar and Chandrashekhara[6] for the case of axisymmetric cylindrical shells.

In the present work, the earlier work is extended in two important ways. First, the more general case of an unsymmetric shell is allowed; secondly, the computer algorithm to be developed will allow the derivation of thick shell theories of arbitrarily high order.

The present work addresses itself to isotropic circular cylindrical shells, although it will be seen that the more general case of anisotropic materials would follow along similar lines, but with more involved algebra.

For circular cylindrical shells, three-dimensional elasticity equations are solved by assuming Taylor series expansions, in the radial direction for stresses and displacements, making use of a convenient point within or at the edge of the shell. The formulation becomes exact when an infinite number of terms are used in the Taylor series expansions.

In practice, only a finite number of terms can be used, and the number of terms to be retained depends on the accuracy desired for a given ratio of thickness to radius (h/a). For thick shells, it is necessary to retain more terms.

Depending upon the number of terms retained, one recovers the classical theories (referred to later as eighth-order), the shear deformation-transverse normal stress theories (referred to as tenth-order), and higher order theories (referred to as twelfth-order, etc.).

In each case, by carrying out the symbolic algebra using the digital computer, it is possible to derive partial differential equations for the different shell theories of arbitrarily high order (only limited by the storage capability of the computer).

One can recover the associated boundary conditions for any given problem to the same order of accuracy as used in the derivation of the differential equation.

Thus, using this method with its associated computer program, one can realize a spectrum of approximate shell theories ranging from the classical thin shell, through all current thick shell theories, right up to three-dimensional elastic theories.

In many problems (i.e. involving stress concentrations and other problems where analytic solutions are preferred to numerical solutions found by finite element or finite difference methods), this method is of particular value for deriving suitable differential equations and boundary conditions for a shell with a given h/a.

In order to demonstrate the comparative accuracy of the various theories, numerical comparisons are made between the current theories and various shell theories, as well as the exact (three-dimensional) theory.

Analysis

Starting from three-dimensional elasticity equations, particular relations among stresses and displacements and their derivatives are found in matrix form. Then, by successive differentiations of these equations, relations between stresses and displacements and higher derivatives are found.

Using a matrix notation and substituting these derivatives into Taylor series expansions at a given point along the radius (exterior or interior edges or mid-point) yields a formal series solution of the elasticity equations.

The differential equations of equilibrium and strain components of an isotropic elastic body in cylindrical coordinates r, θ, z with corresponding displacement components u, v, w in the general case are:

$$\frac{\partial\bar{\sigma}_r}{\partial\bar{r}} + \frac{1}{\bar{r}}\frac{\partial\bar{\tau}_{r\theta}}{\partial\theta} + \frac{\partial\bar{\tau}_{rz}}{\partial z} + \frac{\bar{\sigma}_r - \bar{\sigma}_\theta}{\bar{r}} = 0$$

$$\frac{\partial\bar{\tau}_{r\theta}}{\partial\bar{r}} + \frac{1}{\bar{r}}\frac{\partial\bar{\sigma}_\theta}{\partial\theta} + \frac{\partial\bar{\tau}_{\theta z}}{\partial z} + 2\frac{\bar{\tau}_{r\theta}}{\bar{r}} = 0 \tag{1}$$

$$\frac{\partial\bar{\tau}_{rz}}{\partial\bar{r}} + \frac{1}{\bar{r}}\frac{\partial\bar{\tau}_{\theta z}}{\partial\theta} + \frac{\partial\bar{\sigma}_z}{\partial z} + \frac{\bar{\tau}_{rz}}{\bar{r}} = 0$$

$$\gamma_{r\theta} = \frac{1}{\bar{r}}\frac{\partial\bar{u}}{\partial\theta} + \frac{\partial\bar{v}}{\partial\bar{r}} - \frac{\bar{v}}{\bar{r}}; \quad \epsilon_r = \frac{\partial\bar{u}}{\partial\bar{r}}$$

$$\gamma_{rz} = \frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial\bar{r}}; \quad \epsilon_\theta = \frac{\bar{u}}{\bar{r}} + \frac{\partial\bar{v}}{\bar{r}\partial\theta} \tag{2}$$

$$\gamma_{z\theta} = \frac{\partial\bar{v}}{\partial z} + \frac{1}{\bar{r}}\frac{\partial\bar{w}}{\partial\theta}; \quad \epsilon_z = \frac{\partial\bar{w}}{\partial z}$$

and, according to the generalized Hooke's law for an isotropic material,†

$$\epsilon_r = \frac{1}{E}[\bar{\sigma}_r - \nu(\bar{\sigma}_\theta + \bar{\sigma}_z]; \quad \gamma_{r\theta} = \frac{\bar{\tau}_{r\theta}}{G}$$

$$\epsilon_\theta = \frac{1}{E}[\bar{\sigma}_\theta - \nu(\bar{\sigma}_r + \bar{\sigma}_z)]; \quad \gamma_{rz} = \frac{\bar{\tau}_{rz}}{G} \tag{3}$$

$$\epsilon_z = \frac{1}{E}[\bar{\sigma}_z - \nu(\bar{\sigma}_r + \bar{\sigma}_\theta)]; \quad \gamma_{z\theta} = \frac{\bar{\tau}_{z\theta}}{G}.$$

For cylindrical layer studies, we can reduce eqns (1), (2), and (3) from a system of 15 equations to a system of six first-order equations with respect to the r derivatives regarding $\partial/\partial\theta$, $\partial/\partial z$ as formal symbolic operators.

In matrix form, the six equations become

$$[Y]' = [S][Y], \tag{4}$$

where

$$[S] = \begin{bmatrix} \left(\frac{\nu P - 1}{r}\right) & -\alpha & -\beta/r & \frac{2P}{r^2} & \frac{2P\beta}{r^2} & \frac{2\nu P\alpha}{r} \\ -\alpha\nu P & -\frac{1}{r} & 0 & -\frac{2\nu P\alpha}{r} & -\alpha\beta P\left(\frac{1+\nu}{r}\right) & -\left(2P\alpha^2 + \frac{\beta^2}{r^2}\right) \\ -\nu P\frac{\beta}{r} & 0 & -\frac{2}{r} & \frac{2P\beta}{r^2} & -\left(\alpha^2 + \frac{2P\beta^2}{r^2}\right) & -\alpha\beta P\left(\frac{1+\nu}{r}\right) \\ \left(\frac{1}{2} - \nu\right)P & 0 & 0 & -\frac{\nu P}{r} & -\frac{\nu P\beta}{r} & -\nu P\alpha \\ 0 & 0 & 1 & -\frac{\beta}{r} & \frac{1}{r} & 0 \\ 0 & 1 & 0 & -\alpha & 0 & 0 \end{bmatrix}$$

† This method can also be used for anisotropic materials with the same algebraic procedure.

and

$$[Y] = \begin{bmatrix} \sigma_r \\ \tau_{rz} \\ \tau_{r\theta} \\ u \\ v \\ w \end{bmatrix}; \qquad [Y]' = \begin{bmatrix} \sigma_r' \\ \tau_{rz}' \\ \tau_{r\theta}' \\ u' \\ v' \\ w' \end{bmatrix},$$

with $(\ )' = a\dfrac{\partial}{\partial r}(\ )$ remaining stresses can be obtained from

$$\begin{aligned} \sigma_z &= \frac{2\nu P}{r}u + \frac{2\nu P}{r}\beta v + 2P\alpha w + \nu P\sigma_r \\ \sigma_\theta &= \frac{2P}{r}u + \frac{2P\beta}{r}v + 2P\alpha w + \nu P\sigma_r \\ \tau_{z\theta} &= \alpha v + \frac{\beta}{r}w. \end{aligned} \tag{5}$$

Higher derivatives of $[Y]'$ can be obtained from eqn (4) by successive differentiations. (Note: It is convenient to write $S_{ij} = \overset{(1)}{S_{ij}}$) then

$$\begin{aligned} y_i' &= \sum \overset{(1)}{S_{ij}}\, y_j \\ y_i'' &= \sum (\overset{(1)'}{S_{ij}}\, y_j + \overset{(1)}{S_{ij}}\, y_j') \\ y_i'' &= \sum (\overset{(1)'}{S_{ik}} + \overset{(1)(1)}{S_{ij}S_{jk}})\, y_k, \end{aligned} \tag{6}$$

or

$$y_i'' = \sum \overset{(2)}{S_{ik}}\, y_k \tag{7}$$

where

$$\overset{(2)}{S_{ik}} = \overset{(1)'}{S_{ik}} + \overset{(1)(1)}{S_{ij}\,S_{jk}},$$

etc. Now for the nth derivative

$$\overset{n}{[Y]} = \overset{(n)}{[S]}\,[Y] \tag{8}$$

In eqn (8) $\overset{(n)}{[S]}$ is a matrix of differential operators α, β for any order of derivation. For any given Poisson's ratio ν and any radius r in radial direction ranging from 1 to $1 + \mu$, computer programs‡ give elements of $\overset{(n)}{[S]}$ up to $n = 7$ as function of α, β.

‡ Copies of these programs are given in [7].

HIGHER ORDER THEORIES FOR THICK CYLINDRICAL SHELLS

Now, by making use of shell boundary conditions in the radial direction, we derive partial differential equations for cylindrical shells ranging from classical through current thick elastic shell theories, as well as new higher-order shell theories.

In order to compare the various theories with the exact theory, we are going to study, as a special case, a circular cylinder with no loading on the interior surface and a given pressure on the exterior surface§, i.e. at

$$r = 1 ; \quad \sigma_r = \tau_{rz} = \tau_{r\theta} = 0$$

$$r = 1 + \mu ; \quad \sigma_r = -q, \tau_{rz} = \tau_{r\theta} = 0.$$

It may be noted that other classes of practically important problems can be studied along very similar lines, e.g. a cylindrical elastic layer with a stress-free outer surface and bonded to a rigid cylindrical shaft at $r = 1$ would have boundary conditions: at

$$r = 1 ; \quad u = v = w = 0$$

$$r = 1 + \mu ; \quad \sigma_r = \tau_{r\theta} = \tau_{rz} = 0$$

Taylor series expansions

In order to derive partial differential equations governing the various theories, we use Taylor series expansions at the interior ($r = 1$) or exterior edge ($r = 1 + \mu$), by using the notation previously developed.

Expanding about the interior edge:

$$[Y(r)] = [Y(1)] + (r - 1) [\overset{(1)}{S}(1)] [Y(1)] + \frac{(r-1)^2}{2!} [\overset{(2)}{S}(1)] [Y(1)] + \dots$$

or

$$[Y(r)] = [Y(1)] + \left\{ \sum_{m=1}^{M} \frac{(r-1)^m}{m!} [\overset{(m)}{S}(1)] \right\} [Y(1)]. \tag{9}$$

Expanding about the exterior edge:

$$[Y(r)] = [Y(b/a)] + \left\{ \sum_{m=1}^{M} \frac{(r - b/a)^m}{m!} [\overset{(m)}{S}(b/a)] \right\} [Y(b/a)]. \tag{10}$$

Derivation of equations for shells with an inner surface free of loading, and an outer surface under pressure

For a cylinder with no loading on the inner surface, and pressure acting on the outer surface, we have:

$$\begin{aligned} r = 1 ; \quad & \sigma_r = \tau_{rz} = \tau_{r\theta} = 0 \\ r = 1 + \mu ; \quad & \sigma_r = -q, \tau_{rz} = \tau_{r\theta} = 0. \end{aligned} \tag{11}$$

Now, by applying the above boundary conditions to eqn (9), we find:

$$[Y(b/a)] = [Y(1)] + \left\{ \sum_{m=1}^{M} \frac{\mu^m}{m!} [\overset{(m)}{S}(1)] \right\} [Y(1)] \tag{12}$$

§ The more general case of three stress components presented on both edges would follow along similar line.

or

$$\begin{bmatrix} \sigma_r\left(\frac{b}{a}\right) \\ \tau_{rz}\left(\frac{b}{a}\right) \\ \tau_{r\theta}\left(\frac{b}{a}\right) \\ u\left(\frac{b}{a}\right) \\ v\left(\frac{b}{a}\right) \\ w\left(\frac{b}{a}\right) \end{bmatrix} = \begin{bmatrix} -q \\ 0 \\ 0 \\ u\left(\frac{b}{a}\right) \\ v\left(\frac{b}{a}\right) \\ w\left(\frac{b}{a}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u(1) \\ v(1) \\ w(1) \end{bmatrix} + \begin{bmatrix} \overset{(m)}{T}_{11} & \overset{(m)}{T}_{12} & \overset{(m)}{T}_{13} & \overset{(m)}{T}_{14} & \overset{(m)}{T}_{15} & \overset{(m)}{T}_{16} \\ \overset{(m)}{T}_{21} & \overset{(m)}{T}_{22} & \overset{(m)}{T}_{23} & \overset{(m)}{T}_{24} & \overset{(m)}{T}_{25} & \overset{(m)}{T}_{26} \\ \overset{(m)}{T}_{31} & \overset{(m)}{T}_{32} & \overset{(m)}{T}_{33} & \overset{(m)}{T}_{34} & \overset{(m)}{T}_{35} & \overset{(m)}{T}_{36} \\ \overset{(m)}{T}_{41} & \overset{(m)}{T}_{42} & \overset{(m)}{T}_{43} & \overset{(m)}{T}_{44} & \overset{(m)}{T}_{45} & \overset{(m)}{T}_{46} \\ \overset{(m)}{T}_{51} & \overset{(m)}{T}_{52} & \overset{(m)}{T}_{53} & \overset{(m)}{T}_{54} & \overset{(m)}{T}_{55} & \overset{(m)}{T}_{56} \\ \overset{(m)}{T}_{61} & \overset{(m)}{T}_{62} & \overset{(m)}{T}_{63} & \overset{(m)}{T}_{64} & \overset{(m)}{T}_{65} & \overset{(m)}{T}_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ u(1) \\ v(1) \\ w(1) \end{bmatrix} \tag{13}$$

where

$$\overset{(m)}{T}_{ij} = \sum^{m} \frac{\mu^m}{m!} \overset{(m)}{S}_{ij}(1)$$

or, because the present problem involves stress boundary conditions at b/a, we write:

$$\begin{bmatrix} \overset{(m)}{T}_{34} & \overset{(m)}{T}_{35} & \overset{(m)}{T}_{36} \\ \overset{(m)}{T}_{24} & \overset{(m)}{T}_{25} & \overset{(m)}{T}_{26} \\ \overset{(m)}{T}_{14} & \overset{(m)}{T}_{15} & \overset{(m)}{T}_{16} \end{bmatrix} \begin{bmatrix} u(1) \\ v(1) \\ w(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q \end{bmatrix}, \tag{14}$$

which can be rewritten in the form:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u(1) \\ v(1) \\ w(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -q \end{bmatrix}. \tag{15}$$

Let

$$\overset{(m)}{a}_{ij} = \frac{1}{m!} \overset{(m)}{S}_{kl}(1)$$

where

$$k = 4 - i; \qquad l = j + 3,$$

therefore

$$A_{ij} = \sum_{1}^{m} \mu^m \overset{(m)}{a}_{ij}, \tag{16}$$

or with $[U(1)]^T = [u(1), v(1), w(1)]$ and $[Q]^T = [0, 0, -q]$

$$[A]\,[U(1)] = [Q]. \tag{17}$$

We note that u, v, w are functions of r, θ, z, while $u(1)$, $v(1)$, $w(1)$ are function of θ and z only, and $[A]$ is a matrix of differential operators.

Equation (17) represents a set of three partial differential equations. Nontrivial homogeneous solutions of (17) involve an equation of the form:

$$|A|F = 0, \tag{18}$$

where F is a convenient auxiliary function.

In practice, the number of terms to be retained in eqns (9) or (10) in the Taylor series expansions depends on the ratio of thickness to radius (h/a) and the accuracy desired.

In the expansions, if one retains terms up to third derivative, then the result $|A|F = 0$ will be called the eighth-order shell theory; then, by adding fourth derivative terms, the result will be called the improved eighth-order shell theory; and then by adding fifth up to eighth derivative terms, the results will be called tenth-, improved tenth-, twelfth-, and improved twelfth-order shell theories.

By comparing the results of $|A|F = 0$ for each case, one notices that by adding the fourth derivative terms, the coefficients of what we call the eighth-order shell theory have been stabilized, which means adding the fifth, sixth, seventh, and eighth higher derivatives does not change the coefficients of terms in the eighth-order theory.

Also by adding the sixth derivative terms, the coefficients of the tenth-order theory have been stabilized, and by adding the eighth derivatives, the coefficients of twelfth-order theory have been stabilized.

The following table shows the different order shell theories corresponding to the number of derivative terms retained in the Taylor series expansion.

Computer programs which produce, respectively, elements of $[A]$ and the coefficients of $|A|$ as functions of α, β, and μ for expansion at interior edge ($r = 1$) for a given ν for different orders eighth- up to improved twelfth-order, are given in [7].

As a check on the computer algorithms up to the fourth derivative, elements of $[A]$ have been derived by hand and are given in Table 2. They are given as a function of ν, whereas the computer programs are written to accept a given numerical value for ν.

For the eighth- or improved eighth-order, expanding the determinant after dividing by $\mu^5 P^2/3$ gives:

$$\left(\frac{3}{\mu^5 P^2}\right)|A| = \nabla^8 + 6(1-\nu^2)\,\alpha^6 + 2\beta^6 + 8\alpha^2\beta^4 + 6\nu^2\alpha^4\beta^2$$
$$+ \alpha^4(1-\nu^2)\left[\frac{12}{\mu^2} - \frac{30}{\mu} + 63\right] + \beta^4 + 4\alpha^2\beta^2 + G_s = 0. \tag{19}$$

Where G_s represents terms in (19) of μ and higher order.

As mentioned before, for a given ν, computer programs give coefficients of all terms in $|A|$ which include G_s.

Table 1. Relation of the number of derivative terms to the order of the shell theory

No. of Derivative Terms = IX	Order of Shell Theory = IO
3	Eighth-order
4	Improved eighth-order
5	Tenth-order
6	Improved tenth-order
7	Twelfth-order
8	Improved twelfth-order

Table 2. Elements of $[A]$ up to fourth derivatives as functions of α, β, μ, ν

$$A_{11} = \mu\{-2P\beta\} + \mu^2/2\{2P\beta^3 + 2P\alpha^2\beta + 8P\beta\} + \mu^3/6\{-4P\alpha^2\beta(1+\nu) - 8P\beta^3 - 38P\beta\} + \mu^4/24\{-4P\beta^5 - 4P\alpha^4\beta - 8P\alpha^2\beta^3 + 30P\beta^3 + 2\alpha^2\beta P(12\nu + 11) + 214P\beta\}$$

$$A_{12} = \mu\{-\alpha^2 - 2P\beta^2\} + \mu^2/2\{\alpha^2 + 6P\beta^2\} + \mu^3/6\{\alpha^4 + 4P\beta^4 + \alpha^2\beta^2P(5-\nu) - 6\alpha^2 - 26P\beta^2\} + \mu^4/24\{-2\alpha^4 - 40P\beta^4 + 6\alpha^2\beta^2P(5-\nu) + 30\alpha^2 + 140P\beta^2\}$$

$$A_{13} = \mu\{-\alpha\beta P(1+\nu)\} + \mu^2/2\{\alpha\beta P(5\nu + 3\} + \mu^3/6\{\alpha^3\beta P(\nu+3) + \alpha\beta^3P(\nu+3) - 2\alpha\beta P(6 + 13\nu)\} + \mu^4/24\{-2\alpha\beta^3P(15 + 7\nu) - 2\alpha^3\beta P(9 + 5\nu) + 2\alpha\beta P(30 + 77\nu)\}$$

$$A_{21} = \mu\{-2\nu P\alpha\} + \mu^2/2\{2P\alpha^3 + 2P\alpha\beta^2 + 2\alpha\nu P\} + \mu^3/6\{-2\alpha^3 - 6\alpha\beta^2 - 6\nu P\alpha\} + \mu^4/24\{-4P\alpha^5 - 4P\alpha\beta^4 - 8P\alpha^3\beta^2 + 2\alpha^3P(5 - 2\nu) + 2\alpha\beta^2P(13 - 18\nu) + 24\nu P\alpha\}$$

$$A_{22} = \mu\{-\alpha\beta P(1+\nu)\} + \mu^2/2\{\alpha\beta/2\} + \mu^3/6\{\alpha^3\beta P(\nu+3) + \alpha\beta^3P(\nu+3) + \alpha\beta P(\nu-3)\} + \mu^4/24\{2P\alpha^3\beta(\nu-3) - 2\alpha\beta^3P(9+\nu) + 2P\alpha\beta(6-\nu)\}$$

$$A_{23} = \mu\{-2P\alpha^2 - \beta^2\} + \mu^2/2\{3\beta^2 + 2P\alpha^2\} + \mu^3/6\{4P\alpha^4 + \beta^4 + \alpha^2\beta^2P(5-\nu) - 6P\alpha^2 - 11\beta^2\} + \mu^4/24\{-8P\alpha^4 - 10\beta^4 + 6\alpha^2\beta^2P(\nu-5) + 24P\alpha^2 + 50\beta^2\}$$

$$A_{31} = \mu\{2P\} + \mu^2/2\{-6P\} + \mu^3/6\{-2P\alpha^4 - 2P\beta^4 - 4P\alpha^2\beta^2 + 4\nu P\alpha^2 - 8P\beta^2 + 24P\} + \mu^4/24\{4P\alpha^4 + 20P\beta^4 + 24P\alpha^2\beta^2 - 24\nu P\alpha^2 + 80P\beta^2 - 120P\}$$

$$A_{32} = \mu\{2P\beta\} + \mu^2/2\{2P\beta\alpha^2 + 2P\beta^3 - 4P\beta\} + \mu^3/6\{-16P\beta^3 + 4\alpha^2\beta P(\nu-2) + 14P\beta\} + \mu^4/24\{-4P\beta^5 - 8P\alpha^2\beta^3 - 4P\alpha^4\beta + 110P\beta^3 + 2\alpha^2\beta P(23 - 12\nu) - 66P\beta\}$$

$$A_{33} = \mu\{2\nu P\alpha\} + \mu^2/2\{2P\alpha^3 + 2P\alpha\beta^2 - 6\nu P\alpha\} + \mu^3/6\{-2P\alpha^3(\nu+2) - 6\alpha\beta^2P\cdot(\nu+2) + 24\nu P\alpha\} + \mu^4/24\{-4P\alpha^5 - 4P\alpha\beta^4 - 8P\alpha^3\beta^2 + 6\alpha^3P(3+2\nu) + 2P\alpha\beta^2\cdot(37 + 30\nu) - 120\nu P\alpha\}$$

A class of homogeneous solutions

In order to compare the present higher-order shell theories with some of the previously-derived shell theories and with the elasticity theory, one could solve a specific boundary value problem and compare the predicted stresses and displacements. In the interest of comparing the theories for a whole class of practical shell problems, we take a particular class of homogeneous solutions which are used in books on shell theory to study bending effects in shells[8]. For closed cylindrical shells, solutions that are periodic in θ are useful, since the various cylindrical shell theories have constant coefficients, as seen in (19). A useful class of solutions is of the form:

$$\begin{aligned} u(1) &= \sum A_{\lambda n} e^{\lambda z} \cos n\theta \\ v(1) &= \sum B_{\lambda n} e^{\lambda z} \sin n\theta \\ w(1) &= \sum C_{\lambda n} e^{\lambda z} \cos n\theta, \end{aligned} \tag{20}$$

where λ is to be determined.

Now, if we substitute the above expressions into eqn (15) for different cases (eighth-order up to improved twelfth-order), we will get, in each case, a set of three simultaneous equations for $A_{\lambda n}$, $B_{\lambda n}$, $C_{\lambda n}$ which will be satisfied, provided that:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} A_{\lambda n} \\ B_{\lambda n} \\ C_{\lambda n} \end{bmatrix} = 0. \tag{21}$$

For nontrivial solutions, $|L|$ must vanish, which results in an equation for λ and n for given μ and ν values.

For the eighth-order case:

$$\begin{aligned} P_8 = {} & \underline{12\,\lambda^4(1+\nu)} + \mu^2 P\,[\underline{(\nu^2 - n^2)^4} - \underline{2n^6} - 6(1-\nu^2)\lambda^6 \\ & - 6\nu^2\lambda^4 n^2 + \underline{8\lambda^2 n^4} + 63\,\lambda^4(1-\nu^2) + \underline{n^4} \\ & - \underline{4\lambda^2 n^2}] - 30\,\lambda^4(1+\nu)\,\mu = 0. \end{aligned} \tag{22}$$

Terms of higher order than μ^2 have been neglected in above equation, because μ is small for thin shells studied with the eighth-order theory.

For a given μ, the corresponding equation from the Flügge[8] and Sanders[9] theories are

$$\begin{aligned} P_{8,\text{Flügge}} &= \underline{12\lambda^4(1+\nu)} + \mu^2 P\,[\underline{(\lambda^2-n^2)^4} - \underline{2n^6} + 2\nu\lambda^6 \\ &\quad - \underline{6\lambda^4 n^2} - 2\lambda^2 n^4(\nu-4) + \lambda^4(4-3\nu^2) + \underline{n^4} \\ &\quad - 2\lambda^2 n^2(2-\nu)] = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} P_{8,\text{Sanders}} &= \underline{12\lambda^4(1+\nu)} + \mu^2 P\,[\underline{(\lambda^2-n^2)^4} - \underline{2n^6} \\ &\quad - \underline{6\lambda^4 n^2} + \underline{8\lambda^2 n^4} + \underline{n^4} \\ &\quad - \underline{4\lambda^2 n^2}] = 0, \end{aligned} \tag{24}$$

where the underlined terms in eqns (22–24) indicate terms that are in common to pairs of the above theories.

In comparing the three theories, we see that the zeroth-order μ^0 term, $12\lambda^4(1+\nu)$, appears in all three theories, as well as the eighth-order terms in λ and n.

The lower-order terms in λ and n, i.e. sixth- and fourth-order, differ for the three theories. This means that, for larger λ or n, they would all asymptotically give the same roots.

Computer programs¶ calculate elements of $[L]$ and coefficients of $|L| = 0$ as a function of λ, n, μ for a given ν for eighth-order up to improved twelfth-order theories.

SOME NUMERICAL RESULTS FOR "END SOLUTIONS" OF HIGHER ORDER SHELL THEORIES

We are going to calculate roots of polynominals which provide solutions for different-order shell theories. These solutions will be compared with solutions from classical shell theories and three-dimensional elasticity solutions.

Roots of characteristic polynominals

Eighth-order theory. From eqn (21) we see that eighth- or improved eighth-order theory leads to fourth degree equations for λ^2.

We will see that its four roots are two pairs of conjugate complex roots, and therefore will have four pairs of conjugate complex numbers for λ-roots.

The eight roots may be written in the following form

$$\begin{aligned} \lambda_1 &= -x_1 + iy_1, & \lambda_5 &= x_1 + iy_1 \\ \lambda_2 &= -x_1 - iy_1, & \lambda_6 &= x_1 - iy_1 \\ \lambda_3 &= -x_2 + iy_2, & \lambda_7 &= x_2 + iy_2 \\ \lambda_4 &= -x_2 - iy_2, & \lambda_8 &= x_2 - iy_2, \end{aligned} \tag{25}$$

where the x_k, y_k are real.

Each of the eight values λ; yields in independent solution of (15).

Tenth-order theory. In the case of tenth- or improved tenth-order theory, we must have at least one real root for λ^2, since we have a fifth-order polynomial in λ^2 with real coefficients; therefore, since it turns out that the real λ^2-root is positive, we get

$$\lambda_9 = x_3; \qquad \lambda_{10} = -x_3, \tag{26}$$

where x_3 is real.

¶ See [7].

Table 3. Exponential coefficients for eighth-order and improved eighth-order theories (ν = .3). n = 3, 5 and h/r_0 = .01, .1

h/r_0	n / λ	Eighth-order 3	Eighth-order 5	Improved eighth-order 3	Improved eighth-order 5
.01	x_1	.331273	.990640	.335941	1.00498
	y_1	.315894	.860523	.320096	.871050
	x_2	13.2272	13.8865	13.1977	13.8663
	y_2	12.4340	11.8896	12.3962	11.8457
.1	x_1	.7538	2.0543	1.1052	2.8489
	y_1	.5210	.8486	.7317	1.2391
	x_2	5.1083	6.6279	5.1458	6.8659
	y_2	3.1586	2.6185	2.8910	2.4002

Table 4. Exponential coefficients for tenth-order and improved tenth-order theories (ν = .3). n = 3, 5 and h/r_0 = .01, .1, .2

h/r_0	n / λ	Tenth-order 3	Tenth-order 5	Improved tenth-order 3	Improved tenth-order 5
.01	x_1	.335896	1.00497	.335898	1.00498
	y_1	.320052	.8700998	.320054	.871004
	x_2	13.1668	13.8355	13.1666	13.8353
	y_2	12.4285	11.8779	12.4286	11.8780
	x_3	182.007	182.050	181.684	181.728
	y_3	0	0	0	0
.1	x_1	1.0727	2.7723	1.0840	2.8092
	y_1	.7118	1.1970	.7165	1.2011
	x_2	5.0731	6.7928	5.0631	6.7837
	y_2	3.0285	2.5599	3.0170	2.5322
	x_3	17.980	18.225	17.556	17.983
	y_3	0	0	0.	0
.2	x_1	1.1748	2.3521	1.4103	3.2523
	y_1	.6388	.5298	.7345	1.0801
	x_2	4.1164	5.7731	4.109	5.8872
	y_2	1.8210	1.277	1.6588	1.3022
	x_3	8.8143	8.98936	8.6874	9.5811
	y_3	0	0	0	0

Table 5. Exponential coefficients for twelfth-order and improved twelfth-order theories (ν = .3). n = 3, 5 and h/r_0 = .01, .1, .2

h/r_0	n / λ	Twelfth-order 3	Twelfth-order 5	Improved twelfth-order 3	Improved twelfth-order 5
.01	x_1	.335898	1.00498	.335898	1.00498
	y_1	.320054	.871004	320054	.871004
	x_2	13.1668	13.8355	13.1668	13.8355
	y_2	12.4287	11.8781	12.4287	11.8781
	x_3	208.061	208.094	207.990	208.023
	y_3	78.6954	78.6830	78.7330	78.7204
.1	x_1	1.08222	2.8026	1.0825	2.8038
	y_1	.7157	1.1995	.7158	1.1996
	x_2	5.0697	6.7916	5.0695	6.7912
	y_2	3.0216	2.5373	3.0215	2.5371
	x_3	20.148	20.443	20.026	20.341
	y_3	7.4184	7.3465	7.459	7.345
.2	x_1	1.3441	3.0260	1.3666	3.1290
	y_1	.7112	1.0232	.7185	1.0424
	x_2	4.1265	5.88356	4.1232	5.88255
	y_2	1.7089	1.39250	1.6982	1.3638
	x_3	9.8563	10.1911	9.7259	10.330
	y_3	3.5541	3.5911	3.4505	3.2678

Twelfth-order theory. In the case of twelfth- or improved twelfth-order theory, it turns out that we have three pairs of conjugate complex roots λ^2, which give four more complex roots, eight of which are of the form given in eqn (25), along with

$$\lambda_9 = x_3 + iy_3, \qquad \lambda_{11} = -x_3 - iy_3$$
$$\lambda_{10} = -x_3 + iy_3, \qquad \lambda_{12} = x_3 - iy_3. \tag{27}$$

Corresponding to each root, we have a homogeneous solution for the corresponding shell theory. These can be superimposed to solve end problems where shells are subjected to edge forces and moments. Computer programs[7] give λ_j for eighth- up to improved twelfth-order, for a given ν, by inputting n, μ.

Numerical comparisons

The following tables show roots of polynominals for $n = 3, 5$ and $h/r_0 = .01, .1, .2$ ($h/a = h/r_0/(1 - .5\ h/r_0)$) for present eighth- up to improved twelfth-order theories (Tables 3, 4, and 5).

Results for Love's first approximation type theory, such as the theory given by Flügge[8], are given in Table 6.

Table 6. Exponential coefficients ($\nu = .3$), $n = 3, 5$ and $h/r_0 = .01, .1$

h/r_0	n / λ	3	5
.01	x_1	.337750	1.01058
	y_1	.321524	.875004
	x_2	13.20597	13.8784
	y_2	12.5183	11.9652
.1	x_1	1.14818	2.97213
	y_1	.748438	7.08014
	x_2	5.25693	7.08014
	y_2	3.27008	2.77926

Exponential coefficients have been calculated from Flügge[8].

Table 7. Exponential coefficients ($\nu = .3$), $n = 3, 5$ and $h/r_0 = .01, .1, .2$

h/r_0	n / λ	3	5
.01	x_1	.337725	1.01037
	y_1	.321503	.874888
	x_2	13.2363	13.9086
	y_2	12.4880	11.9350
	x_3	316.247	316.272
	y_3	0	0
.1	x_1	1.1407	2.95078
	y_1	.7484	1.25401
	x_2	5.3454	7.1546
	y_2	3.1744	2.6694
	x_3	31.818	32.071
	y_3	0	0
.2	x_1	1.5159	3.4604
	y_1	.7871	1.1423
	x_2	4.5864	6.52985
	y_2	1.8836	1.52634
	x_3	16.2083	16.7152
	y_3	0	0

Exponential coefficients have been calculated from Naghdi and Cooper[10].

Table 8. Exponential coefficients for the elasticity solution ($\nu = .3$), $n = 3, 5$ and $h/r_0 = .01, .1, .2$

h/r_0	n / λ	3		5	
.01	x_1	.338353	(71)	1.012656	(71)
	y_1	.322173		.875924	
	x_2	13.1151	(121)	13.7825	(111)
	y_2	12.3741		11.8168	
	x_3	310.539	(151)	310.564	(151)
	y_3	0		0	
.1	x_1	1.08868	(71)	2.8162	(71)
	y_1	.716860		1.1959	
	x_2	5.05255	(91)	6.7690	(71)
	y_2	2.99760		2.5056	
	x_3	29.7129	(91)	29.9575	(91)
	y_3	0		0	
.2	x_1	1.3738	(31)	3.1200	(71)
	y_1	.7157		1.0365	
	x_2	4.108	(51)	5.86889	(101)
	y_2	1.6703		1.35463	
	x_3	14.3856	(111)	14.8506	(111)
	y_3	0		0	

Numbers in circles indicate number of steps in numerical method.

Table 9. Comparison of exponential coefficients between current theories and classical shell theories as well as the three-dimensional theory for $\nu = .3$, $n = 3$, $h/r_0 = .01$

Theory	x_2	$\Delta x_2 \times 10^{-3}$	% Error in x_2	y_2	$\Delta y_2 \times 10^{-3}$	% Error in y_2
Eighth	13.2272	−112	.85	12.4340	−60	.48
Flügge[1]	13.20597	−91	.69	12.5183	−144	1.15
Improved eighth	13.1977	−82	.63	12.3962	−22	.18
Tenth	13.1668	−52	.39	12.4285	−54	.44
Naghdi[2]	13.2363	−121	.92	12.4880	−114	.92
Improved tenth	13.1666	−52	.39	12.4286	−55	.44
Twelfth	13.1668	−52	.39	12.4287	−55	.44
Improved twelfth	13.1668	−52	.39	12.4287	−55	.44
Elasticity[3]	13.1151	0	0	12.3741	0	0

[1] Exponential coefficient has been calculated from Flügge[8].
[2] Exponential coefficient has been calculated from Naghdi and Cooper[10].
[3] Calculated using a numerical solution of the elasticity equation by a Runge-Kutta method as contained in a program available from the Civil Engineering Department, Univ. of Mass. (TSHELL81).

Table 10. Comparison of exponential coefficients between current theories and classical shell theories, as well as the three-dimensional theory for $\nu = .3$, $n = 5$, $h/r_0 = .2$

Theory	x_2	$\Delta x_2 \times 10^{-3}$	% Error in x_2	y_2	$\Delta y_2 \times 10^{-3}$	% Error in y_2
Tenth	5.7731	96	1.6	1.277	78	6
Naghdi	6.52985	−660	11	1.52634	−172	13
Improved tenth	5.8872	−18	.3	1.3022	52	4
Twelfth	5.88356	−15	.25	1.39250	−38	2.8
Improved twelfth	5.88255	−14	.23	1.36381	−9	.68
Elasticity	5.86889	0	0	1.35463	0	0

Results for a shear deformation-transverse normal stress type theory, such as has been given by Naghdi and Cooper[10], are given in Table 7, and results of the three-dimensional elasticity theory[||] in Table 8.

For purpose of comparison, in Tables 9–10 the various higher-order theories are compared in terms of the real and imaginary parts of predicted exponential roots for $\begin{cases} n = 3 \\ h/r_0 = .01 \end{cases}, \begin{cases} n = 5 \\ h/r_0 = .2 \end{cases}$. As we notice in Table 9 for thin shells ($h/r_0 = .01$), tenth- to improved twelfth-order theory are about the same and eighth-order theory gives good results compared to higher order theories, and as we expected, eighth-order theory gives good results for thin shells. But in Table 10 for a thick shell ($h/r_0 = .2$) by going from tenth- to improved twelfth-order theory, we get noticeable corrections with respect to the elasticity solution.

In Tables 3–5, by going from the present eighth- to improved twelfth-order theory, we generally get improvement with respect to the elasticity solution, but as we notice, the energy-type-based Naghdi and Cooper[10] theory gives more accurate results for the highest mode compared to elasticity solution, because the example which is chosen is in favor of Naghdi and Cooper's theory. If we would take other examples with loads in radial direction, we would expect the present theories results to improve considerably, as Iyengar and Chandrashekhara[6] showed in their study. They used the method of initial functions for the case of axisymmetric circular cylindrical shell, subjected to periodically spaced band loads.

Based on their numerical comparison with the Naghdi[11] and Reissner[12] theories as well as elasticity solutions, they concluded that their higher order theories (equivalent to present eighth-, improved eighth-, and tenth-order theory for the axisymmetric case) are more accurate for problems with radial loads. So, in our case, which is more general (i.e. variation with θ) and allows for higher-order equations compared to Iyengar and Chandrashekhara's theory, we would also expect very accurate results. Even for the end problem, we could, of course, go to higher-order (e.g. fourteenth-, sixteenth-, etc.) theories to get more accurate results.

REFERENCES

1. V. Z. Vlasov, The method of initial functions in problems of theory of thick plates and shells, *Proc. 9th Intl. Cong. Applied Mech.* **6,** 321–330. Brussels, Belgium (1957).
2. Y. C. Das and A. V. Setlur, Method of initial functions in two-dimensional elastodynamic problems. (*J. Applied Mech.* **37,** 137–140 (1970).
3. H. M. Haydl, Bending of cylindrical shells by initial parameter method (*J. Engng for Industry*), *Trans. ASME Series B,* **93**(3), 845–850 (1971).
4. K. T. Iyengar, K. Chandrashekhara, and V. K. Sebastian, On the analysis of thick rectangular plates, *Ing. Arch.* **43,** 317–330 (1974).
5. Y. C. Das and N. S. V. Rao Kameswara, *A Mixed Method in Elasticity*, Indian Institute of Technology, Kanpur, India (1976).
6. K. T. Iyengar and K. Chandrashekhara, On thick circular cylindrical shells under axisymmetric loading. *Acta Mechanica* **23** (1975).
7. S. Faraji, Method of initial functions for thick shells, dissertation, University of Massachusetts, Amherst, MA (1983).
8. W. Flügge, *Stresses in Shells*, Springer-Verlag, Berlin (1960).
9. J. L. Sanders, Jr., *An Improved First Approximation Theory for Thin Shell*, NASA Technical Report R-24 (1959).
10. P. M. Naghdi and R. M. Cooper, Progagation of elastic waves in cylindrical shells, including the effects of transverse shear and rotatory inertia, *J. Acoustical Soc. America,* **28**(1), 56–63 (1955).
11. P. M. Naghdi, On the theory of thin elastic shells, *Quart. Appl. Math.* **14,** 369–380 (1957).
12. E. Reissner, Stress strain relation in the theory of thin elastic shells, *J. Math. and Physics* **31,** 109–119 (1952).

[||] Calculated using a numerical solution of the elasticity equation by a Runge-Kutta method, as contained in a program available from the Civil Engineering Department, Univ. of Mass. (TSHELL81).